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Quantum discord of Bell cat-states under amplitude damping

M. Daoud[a*] and **R. Ahl Laamara**[b,c †]

[a] *Department of Physics, Faculty of Sciences, University Ibnou Zohr, Agadir, Morocco*

[b] *LPHE-Modeling and Simulation, Faculty of Sciences, University Mohammed V, Rabat, Morocco*

[c] *Centre of Physics and Mathematics, CPM, CNESTEN, Rabat, Morocco*

Abstract

The evolution of pairwise quantum correlations of Bell cat-states under amplitude damping is examined using the concept of quantum discord which goes beyond entanglement. A closed expression of the quantum discord is explicitly derived. We used of the Koashi-Winter relation. A relation which facilitates the optimization process of the conditional entropy. We also discuss the temporal evolution of bipartite quantum correlations under a dephasing channel and compare the behaviors of quantum discord and entanglement whose properties are characterized through the concurrence.

* email: m_daoud@hotmail.com

† email: ahllaamara@gmail.com

1 Introduction

Quantum entanglement plays an important role in the area of quantum information such as quantum teleportation [1], superdense coding [2], quantum key distribution [3], telecloning [4] and many more. It is one of the most fundamental concepts in the description of quantum correlations in multipartite quantum systems and makes possible tasks in quantum information which are impossible without it. Therefore it was important to quantify the amount of quantum correlations in a given bipartite quantum state. Much effort has been devoted to the classification of quantum states into separable and entangled states (see [5, 6, 7] and references therein). Until some time ago, entanglement was usually regarded as the only kind of nonclassical correlation in a composed state. However, there are other nonclassical correlations different from those involved in entanglement which are useful for quantum technology. Hence, to characterize all nonclassical correlations present in a multipartite system, the so-called quantum discord, which goes beyond entanglement, was introduced in [8, 9]. Now, it is commonly accepted that quantum discord constitutes a new resource for quantum computation. Among the evidences of the relevance of the quantum discord, one may quote for instance quantum non-locality without entanglement [10, 11, 12] and the advantages offered in increasing the rapidity of certain computational tasks with separable states in comparison with their classical counterparts [13, 14, 15, 16]. But, it may be noticed that despite increasing evidences for relevance of the quantum discord in information processing tasks, there is no straightforward method to get the analytical form of quantum discord in a given quantum state. This is mainly due to the fact that its evaluation involves an optimization procedure which is in general a hard task to perform. A reliable algorithm to evaluate quantum discord for general two-qubit states is still missing and only few analytical results were obtained for some particular forms of the so-called two-qubit X-states [17, 18, 19, 20, 21, 22]. This class includes the maximally entangled Bell states and Werner states [23]. A closed expression for the discord of arbitrary states remains an important challenge.

On the other hand, quantum optical tools are expected to be useful in the context of quantum information science, especially for communications using qubits over long distance. However, optical qubits suffer from decoherence due to energy loss or photon absorption. To reduce the decoherence effects, encoding qubits in multi-photon optical coherent states seems to be a promising alternative. They are more robust against small levels of photon absorption (see [24]). The photon loss or amplitude damping in a noisy environment can be modeled by assuming that some of field energy and information is lost after transmission through a beam splitter.

Considering this problem and motivated by many works devoted to study entanglement properties in a composite system involving coherent states (for a recent review see [25]), we investigate in this paper a method to describe the evolution of quantum discord in Bell cat-states under amplitude damping. We present an algorithm to calculate the quantum discord. First, we obtain an explicit and simplified expression for the conditional entropy, exploiting the Bloch representation of the density matrix. Then, we combine the purification method and the Koashi-Winter relation [26] to perform easily the optimization of the conditional entropy. This allows us to get a closed form of quantum discord in damped Bell cat-states. Exploiting our algorithm, we examine the dynamical evolution of the system under a dephasing channel to compare the temporal behaviors of quantum discord and entanglement as quantifiers of non-classical correlations.

This paper is organized as follows. Section 2 provides an introduction to quantum discord. We present the main definitions and properties. Section 3 concerns Glauber coherent states superpositions subjected to an amplitude damping channel. In the optical context, amplitude damping can be appropriately modeled by having the signal interact with a vacuum mode in a beam splitter. Having identified the effect of amplitude damping, we study in section 4 the evolution of the quantum correlations of Bell cat-states under amplitude damping. We obtain the explicit form of quantum discord. Section 5 deals with the comparison of the evolution of quantum discord and entanglement under a dephasing channel, using the results of the previous sections. Finally, the last section recalls the main results of our work and suggests further issues deserving to be investigated.

2 Quantum discord: Generalities

For a state ρ^{AB} of a bipartite quantum system composed of particle A and particle B, the quantum discord is defined as the difference between total correlation $I(\rho^{AB})$ and classical correlation $C(\rho^{AB})$. The total correlation is usually quantified by the mutual information I

$$I(\rho^{AB}) = S(\rho^{A}) + S(\rho^{B}) - S(\rho^{AB}), \tag{1}$$

where $\rho^{A(B)} = \mathrm{Tr}_{B(A)}(\rho^{AB})$ is the reduced state of $A(B)$, and $S(\rho)$ is the von Neumann entropy of a quantum state ρ. Suppose that a positive operator valued measure (POVM) measurement is performed on particle A. The set of POVM elements is denoted by $\mathcal{M} = \{M_k\}$ with $M_k \geqslant 0$ and $\sum_k M_k = \mathbb{I}$. We remind that the generalized positive operator valued measurement is not required. Indeed, it has be shown in [27] that for the optimal measurement for the conditional entropy is ensured by projective one. Thus, a projective measurement on the subsystem A project the system into a statistical ensemble $\{p_k^B, \rho_k^B\}$, such that

$$\rho^{AB} \longrightarrow \rho_k^B = \frac{(M_k \otimes \mathbb{I})\rho^{AB}(M_k \otimes \mathbb{I})}{p_k^B} \tag{2}$$

where the von Neumann measurement for subsystem A writes as

$$M_k = U\,\Pi_k\,U^{\dagger} : \quad k = 0, 1\,, \tag{3}$$

with $\Pi_k = |k\rangle\langle k|$ is the projector for subsystem A along the computational base $|k\rangle$, $U \in SU(2)$ is a unitary operator with unit determinant, and

$$p_k^B = \mathrm{tr}\Big[(M_k \otimes \mathbb{I})\rho^{AB}(M_k \otimes \mathbb{I})\Big].$$

The amount of information acquired about particle B is then given by

$$S(\rho^B) - \sum_k p_k^B\ S(\rho_k^B),$$

which depends on measurement $\mathcal{M}$. This dependence can be removed by doing maximization over all the measurements, which gives rise to the definition of classical correlation:

$$\begin{aligned} C(\rho^{AB}) &= \max\nolimits_{\mathcal{M}}\Big[S(\rho^B) - \textstyle\sum_k\ p_k^B\ S(\rho_k^B)\Big] \\ &= S(\rho^B) - \widetilde{S}_{\min} \end{aligned} \tag{4}$$

where $\widetilde{S}_{\min}$ denotes the minimal value of the conditional entropy

$$\widetilde{S} = \sum_k p_k^B \; S(\rho_k^B). \tag{5}$$

Then, the difference between $I(\rho^{AB})$ and $C(\rho^{AB})$ gives the amount of quantum discord in the system

$$D(\rho^{AB}) = I(\rho^{AB}) - C(\rho^{AB}) = S(\rho^A) + \widetilde{S}_{\min} - S(\rho^{AB}). \tag{6}$$

The main difficulty, for several qubits as well as qudits systems, lies in performing the minimization of the conditional entropy. This explains why there is no straightforward algorithm to compute explicitly quantum discord for mixed states. Only partial results are available. They were obtained for some special forms of the so-called X-states [28, 31, 32, 35].

3 Amplitude damping

The beam splitter offers a simple way to probe the quantum nature of electromagnetic field through simple experiments. The study of entangled states has revived interest in this device. Many authors have considered the behavior of quantum states when passed through a beam splitter [36, 39]. Recently, a quantum network of beam splitters was used to create multi-particle entangled states of continuous variables [40] and also multi-particle entangled coherent states [41]. It also provides, as mentioned in the introduction, a simple way to model the amplitude damping related to the absorption of transmitted photons in a noisy channel.

3.1 Fock state inputs

The beam splitter is an optical device with two input and two output ports that, in some sense, governs the interaction of two harmonic oscillators. The input and output boson operators are related by a unitary transformation which is an element of the $SU(2)$ group defined by

$$\mathcal{B}(\theta) = \exp\left[\frac{\theta}{2}\left(a_1^- a_2^+ - a_1^+ a_2^-\right)\right]. \tag{7}$$

The objects a_l^+ and a_l^- $(l = 1, 2)$ are the usual harmonic oscillator ladder operators. The reflection and transmission coefficients

$$t = \cos\frac{\theta}{2}\,, \qquad r = \sin\frac{\theta}{2} \tag{8}$$

are defined in terms of the angle θ. The operator $\mathcal{B}$ is actually acting on the states $|n_1, n_2\rangle$ of the usual two dimensional harmonic oscillator. If the input state is $|n_1, n_2\rangle$, then the $\mathcal{B}$ action leads to the following Fock states superposition

$$\mathcal{B}|n_1, n_2\rangle = \sum_{m_1,m_2} \langle m_1, m_2|\mathcal{B}|n_1, n_2\rangle |m_1, m_2\rangle = \sum_{m_1,m_2} \mathcal{B}_{n_1,n_2}^{m_1,m_2} |m_1, m_2\rangle \tag{9}$$

and in general the output is a two-particle entangled state. On the other hand, the action of the unitary operator $\mathcal{B}$ on the state $|n, 0\rangle$ gives

$$\mathcal{B}|n, 0\rangle = \left(1 + |\xi|^2\right)^{-\frac{n}{2}} \sum_{m=0}^{n} \xi^m \frac{\sqrt{n!}}{\sqrt{(n-m)!m!}} |n - m, m\rangle \tag{10}$$

where the new variable $\xi = r/t$ is defined as the ratio of the reflection and transmission coefficients of the beam splitter under consideration. Then, the output state (10) turns out to be the $SU(2)$ coherent state associated with the unitary representation labeled by the integer n. This method can be extended to a chain of k beam splitters to generate $SU(k+1)$ coherent states labeled by the reflection and transmission parameters (see for instance [42, 43] where similar notations were used). It is important to stress that the generation of coherent states using beam splitters requires input radiation state with fixed number of photons. The experimental production of such interesting and highly non classical states has been investigated during the last decade (see [44] and references therein). Recently an important experimental advance was reported by Hofheinz et al in [45]. They gave the first experimental demonstration for generating photon number Fock states containing up to $n = 6$ photons in a super-conducting quantum circuit.

3.2 coherent state inputs

To describe the photon loss, we usually assume that some of the coherent field is lost in transit via a beam splitter. The coherent states enters one port of the beam splitter and the vacuum, representing the environment, enters the second port. After transmission some information encoded in the coherent states is transferred and the remaining amount of information is lost to the noisy channel. To find the final state after transmission, we should first evaluate the action of the operator $\mathcal{B}(\theta)$ on the transmitted state. Here we shall be interested in the superpositions of the form

$$|Q_\alpha\rangle = \frac{1}{\sqrt{N(\alpha)}}(a|-\alpha\rangle + b|\alpha\rangle) \tag{11}$$

where $|a|^2 + |b|^2 = 1$ and $N(\alpha)$ is a normalization factor given by

$$N(\alpha) = 1 + e^{-2|\alpha|^2}(ab^* + a^*b).$$

So, we start by evaluating the beam splitter action on the bipartite state $|\alpha, 0\rangle$ where the Glauber coherent-state $|\alpha\rangle$ is expressed in the Fock (number) basis as

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{12}$$

This yields

$$\mathcal{B}(\theta)|\alpha, 0\rangle = |\alpha t, \alpha r\rangle. \tag{13}$$

It follows that the action of the operator $\mathcal{B}(\theta)$ on the state $|Q_\alpha\rangle$ gives

$$|Q\rangle_t = \frac{1}{\sqrt{N(\alpha)}}(a|-\alpha t, -\alpha r\rangle + b|\alpha t, \alpha r\rangle), \tag{14}$$

a state describing the quantum field and loss modes (environment). The final state is then obtained by performing a partial trace over the loss mode l. We get

$$\rho = \sum_{n=0}^{\infty} {}_l\langle n|Q\rangle_t \; {}_t\langle Q|n\rangle_l. \tag{15}$$

A straightforward calculation gives

$$\rho = \frac{1+c}{2}\left[\left(\frac{1}{\sqrt{N(\alpha)}}(a|-\alpha t\rangle + b|\alpha t\rangle)\right) \times \text{h.c.}\right] + \frac{1-c}{2}\left[\left(\frac{1}{\sqrt{N(\alpha)}}(a|-\alpha t\rangle - b|\alpha t\rangle)\right)\right] \times \text{h.c.}\right], \quad (16)$$

where h.c. stands for Hermitian conjugation and c is the the coherent states overlapping $c = e^{-2r^2|\alpha|^2}$. The last equation can be also written as

$$\rho = \frac{N(\alpha t)}{N(\alpha)}\left[\frac{1}{2}(1+c)|Q_{\alpha t}\rangle\langle Q_{\alpha t}| + \frac{1}{2}(1-c)Z|Q_{\alpha t}\rangle\langle Q_{\alpha t}|Z\right], \quad (17)$$

where Z is the Pauli Z-operator defined by

$$Z|Q_{\alpha t}\rangle = Z\left[\frac{1}{\sqrt{N(\alpha t)}}(a|\alpha t\rangle + b|\alpha t\rangle)\right] = \frac{1}{\sqrt{N(\alpha t)}}(a|\alpha t\rangle - b|\alpha t\rangle)$$

which produces a phase flip in the qubit basis. This implies that the transmission of quantum information encoded in optical coherent states suffers from two main types of error: the reduction of the amplitude of the coherent state and the phase flip generated by the application of the Pauli Z-operator.

3.3 Two mode coherent state inputs

The above considerations can be extended to two mode coherent states of the form

$$|\chi_{\alpha,\alpha}\rangle = \frac{1}{\sqrt{\mathcal{N}_\alpha}}(\sqrt{\omega}\ |\alpha,\alpha\rangle + e^{i\theta}\sqrt{1-\omega}\ |-\alpha,-\alpha\rangle) \quad (18)$$

where

$$\mathcal{N}_\alpha^2 = 1 + 2\sqrt{\omega(1-\omega)}\cos\theta e^{-4|\alpha|^2}.$$

The action of a beam splitter on the state $|\chi_{\alpha,\alpha}\rangle \otimes |0\rangle$ gives

$$|\chi\rangle_t = |\chi_{\alpha,\alpha}\rangle \otimes |0\rangle = \frac{1}{\sqrt{\mathcal{N}_\alpha}}(\sqrt{\omega}\ |\alpha,\alpha t,\alpha r\rangle + e^{i\theta}\sqrt{1-\omega}\ |-\alpha,-\alpha t,-\alpha r\rangle). \quad (19)$$

The trace over the lost modes gives the density

$$\rho = \text{Tr}_l\ |\chi\rangle_t\ {}_t\langle\chi|. \quad (20)$$

Here again, one can see that the density ρ takes the following compact form

$$\rho = \frac{\mathcal{N}_{\alpha t}}{\mathcal{N}_\alpha}\left[\frac{1}{2}(1+c)|\chi_{\alpha,\alpha t}\rangle\langle\chi_{\alpha,\alpha t}| + \frac{1}{2}(1-c)Z|\chi_{\alpha,\alpha t}\rangle\langle\chi_{\alpha,\alpha t}|Z\right]. \quad (21)$$

The final state is mixed and the amplitude of the second mode is reduced.

4 Evolution of Bell cat-states correlations under amplitude damping

The Bell states are very interesting in quantum optics and have been used in the field of quantum teleportation and many others quantum computing operations. The experimental generation can be realized by sending a cat states of the form $|\sqrt{2}\alpha\rangle + |-\sqrt{2}\alpha\rangle$ and the vacuum into the two input ports of a 50/50 beam splitter $\mathcal{B}(\pi/4)$ (see equation (13)) to get

$$|B_{\alpha,\alpha}\rangle = \frac{1}{\sqrt{N_\alpha}}(|\alpha,\alpha\rangle + |-\alpha,-\alpha\rangle) \quad (22)$$

where the normalization factor is defined by

$$N_\alpha = 2(1 + e^{-4|\alpha|^2}).$$

Obviously, the problem of generating Bell states is deeply dependent on the availability of a source of cat states. For instance, they can be produced by sending a coherent state into a nonlinear medium exhibiting the Kerr effect [46]. They can be also generated using a squeezing interaction, linear optical devices and photon counters [47, 48]. Recently, a promising new method to produce cat states was proposed by Lund *et al.* in [49]. The production of cat states especially ones of high amplitude or mean number of photons remains an experimental challenge. Considering the fast technical progress and the increasing number of groups working in this field, we expect that the generation of cat states (and Bell states) is a goal that is achievable in the near future. The available experimental results in the literature, obtained with the present day technology, are encouraging. Superpositions of weak coherent states with opposite phase, resembling to a small "Schrödinger's cat" state (or "Schrödinger's kitten"), were produced by photon subtraction from squeezed vacuum [50]. Also, the experimental generation of arbitrarily large squeezed cat states, using homodyne detection and photon number states (two photons) as resources was reported in [50]. Very recently, creation of coherent state superpositions, by subtracting up to three photons from a pulse of squeezed vacuum light, is reported in [51]. The mean photon number of such coherent states produced by three-photon subtraction is of 2:75.

4.1 Bell cat-states under amplitude damping.

Using the results of the previous section, it is simple to verify that under the action of a beam splitter on the state (22), the resultant density is

$$\rho^{AB} = \frac{N_{\alpha t}}{N_\alpha}\left[\frac{1}{2}(1+c)|B_{\alpha,\alpha t}\rangle\langle B_{\alpha,\alpha t}| + \frac{1}{2}(1-c)Z|B_{\alpha,\alpha t}\rangle\langle B_{\alpha,\alpha t}|Z\right]. \tag{23}$$

To study the quantum correlations in this state, a qubit mapping is required. It can be defined as follows. For the first mode A, we introduce a two dimensional basis spanned by the vectors $|u_\alpha\rangle$ and $|v_\alpha\rangle$ defined by

$$|\alpha\rangle = a_\alpha|u_\alpha\rangle + b_\alpha|v_\alpha\rangle \qquad |-\alpha\rangle = a_\alpha|u_\alpha\rangle - b_\alpha|v_\alpha\rangle \tag{24}$$

where

$$|a_\alpha|^2 + |b_\alpha|^2 = 1 \qquad |a_\alpha|^2 - |b_\alpha|^2 = \langle -\alpha|\alpha\rangle.$$

To simplify our purpose, we take a_α and b_α reals:

$$a_\alpha = \frac{\sqrt{1+p}}{\sqrt{2}} \quad b_\alpha = \frac{\sqrt{1-p}}{\sqrt{2}} \qquad \text{with} \quad p = \langle -\alpha|\alpha\rangle = e^{-2|\alpha|^2}.$$

Similarly, for the second mode B, a two-dimensional basis generated by the vectors $|u_{\alpha t}\rangle$ and $|v_{\alpha t}\rangle$ is defined as

$$|\alpha t\rangle = a_{\alpha t}|u_{\alpha t}\rangle + b_{\alpha t}|v_{\alpha t}\rangle \qquad |-\alpha t\rangle = a_{\alpha t}|u_{\alpha t}\rangle - b_{\alpha t}|v_{\alpha t}\rangle \tag{25}$$

where

$$a_{\alpha t} = \frac{\sqrt{1+p^{t^2}}}{\sqrt{2}} \quad b_{\alpha t} = \frac{\sqrt{1-p^{t^2}}}{\sqrt{2}}.$$

The density matrix ρ^{AB} (23) can be cast in the following matrix form

$$\rho^{AB} = \frac{2}{N_\alpha} \begin{pmatrix} (1+c)a_\alpha^2 a_{\alpha t}^2 & 0 & 0 & (1+c)a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} \\ 0 & (1-c)a_\alpha^2 b_{\alpha t}^2 & (1-c)a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} & 0 \\ 0 & (1-c)a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} & (1-c)b_\alpha^2 a_{\alpha t}^2 & 0 \\ (1+c)a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} & 0 & 0 & (1+c)b_\alpha^2 b_{\alpha t}^2 \end{pmatrix} \tag{26}$$

in the representation spanned by two-qubit product states

$$|1\rangle = |u_\alpha\rangle_A \otimes |u_{\alpha t}\rangle_B \quad |2\rangle = |u_\alpha\rangle_A \otimes |v_{\alpha t}\rangle_B \quad |3\rangle = |v_\alpha\rangle_A \otimes |u_{\alpha t}\rangle_B \quad |4\rangle = |v_\alpha\rangle_A \otimes |v_{\alpha t}\rangle_B.$$

The visual form of the obtained density (26) resembles the letter X and it is a special kind of the so-called called X-states which have been extensively discussed in the literature [52, 53]. The density ρ^{AB} also rewrites, in the Bloch representation, as

$$\rho^{AB} = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + R_{30}\ \sigma_3 \otimes \mathbb{I} + R_{03}\ \mathbb{I} \otimes \sigma_3 + \sum_{i=1}^{3}\ R_{ii}\ \sigma_i \otimes \sigma_i) \tag{27}$$

where the correlation matrix elements are given by

$$R_{03} = \frac{p^{t^2} + p^{2-t^2}}{1+p^2}, \qquad R_{30} = \frac{2p}{1+p^2}, \tag{28}$$

$$R_{11} = \frac{\sqrt{(1-p^2)(1-p^{2t^2})}}{1+p^2}, \qquad R_{22} = -p^{1-t^2}\frac{\sqrt{(1-p^2)(1-p^{2t^2})}}{1+p^2}, \qquad R_{33} = \frac{p^{1+t^2} + p^{1-t^2}}{1+p^2}. \tag{29}$$

4.2 Quantum mutual information

The density ρ^{AB} is a two qubit state of rank two. The corresponding non vanishing eigenvalues are given by

$$\lambda_1 = \frac{(1+p^{r^2})(1+p^{t^2+1})}{2+2p^2} \qquad \lambda_2 = \frac{(1-p^{r^2})(1-p^{t^2+1})}{2+2p^2}. \tag{30}$$

It follows that the joint entropy is

$$S(\rho^{AB}) = -\lambda_1 \log_2 \lambda_1 - \lambda_2 \log_2 \lambda_2. \tag{31}$$

The quantum mutual information is given by

$$I(\rho^{AB}) = S(\rho^A) + S(\rho^B) + \sum_{j=1,2} \lambda_j \log_2 \lambda_j \tag{32}$$

where ρ^A and ρ^B are the marginal states of ρ^{AB}, and

$$S(\rho^A) = -\lambda_+^A \log_2 \lambda_+^A - \lambda_-^A \log_2 \lambda_-^A \qquad S(\rho^B) = -\lambda_+^B \log_2 \lambda_+^B - \lambda_-^B \log_2 \lambda_-^B \tag{33}$$

with

$$\lambda_\pm^A = \frac{(1 \pm p)^2}{2+2p^2} \qquad \lambda_\pm^B = \frac{(1 \pm p^{t^2})(1 \pm p^{r^2+1})}{2+2p^2}.$$

Reporting (33) into (32), the quantum mutual information reads

$$I(\rho^{AB}) = H\left(\frac{(1+p)^2}{2+2p^2}\right) + H\left(\frac{(1+p^{t^2})(1+p^{r^2+1})}{2+2p^2}\right) - H\left(\frac{(1+p^{r^2})(1+p^{t^2+1})}{2+2p^2}\right) \tag{34}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

4.3 Conditional entropy

After computing the quantum mutual information, we need next to compute the classical correlation $C(\rho^{AB})$ defined by (4). We consider projective measurements for subsystem A. We follow the procedure developed in [54]. We remind that the generalized positive operator valued measurement is not required. Indeed, as mentioned above, the optimal measurement for the conditional entropy is ensured by projective operator [27]. The general form of the $SU(2)$ unitary operator, occurring in (3), is

$$U = \exp(\eta\sigma_+ - \bar{\eta}\sigma_-)\exp(i\phi\sigma_3) \tag{35}$$

where $\eta \in \mathbb{C}$ and $\phi \in \mathbb{R}$. This parametrization allows us to express the quantities defined by

$$\langle\sigma_i\rangle_k = \langle k|U^\dagger\sigma_i U|k\rangle, \qquad i = 1,2,3 \quad \text{and} \quad k = 0,1$$

as

$$\langle\sigma_3\rangle_k = (-)^k\frac{1-\bar{z}z}{1+\bar{z}z}, \qquad \langle\sigma_1\rangle_k = (-)^k\frac{\bar{z}+z}{1+\bar{z}z}, \qquad \langle\sigma_2\rangle_k = i(-)^k\frac{\bar{z}-z}{1+\bar{z}z}$$

where $z = -i\frac{\eta}{\sqrt{\bar{\eta}\eta}}\tan\sqrt{\bar{\eta}\eta}$. They can be also written as

$$\langle\sigma_3\rangle_k = (-)^k\cos\theta, \qquad \langle\sigma_1\rangle_k = (-)^k\sin\theta\cos\varphi, \qquad \langle\sigma_2\rangle_k = (-)^k\sin\theta\sin\varphi$$

where $\frac{\theta}{2}e^{i\varphi} = -i\eta$. These mean values combined with the expressions of the correlation matrix elements (28) and (29) determine the explicit form of the conditional densities (2) and the conditional entropy (5) in terms of the angular variables θ and φ. Indeed, combining the equations (2), (3), (26) and (35), it is simply seen that the density operators ρ_k^B take the following form

$$\rho_k^B = \frac{1}{p_k^B}\begin{pmatrix} (1+R_{03})+(R_{30}+R_{33})\langle\sigma_3\rangle_k & R_{11}\langle\sigma_1\rangle_k - iR_{22}\langle\sigma_2\rangle_k \\ R_{11}\langle\sigma_1\rangle_k + iR_{22}\langle\sigma_2\rangle_k & (1-R_{03})+(R_{30}-R_{33})\langle\sigma_3\rangle_k \end{pmatrix} \tag{36}$$

where

$$p_k^B = \frac{1}{2}(1+R_{30}\ \langle\sigma_3\rangle_k).$$

It follows that the conditional entropy given by (5) rewrites also as

$$\widetilde{S} \equiv \widetilde{S}(\theta,\varphi) = \sum_{k=0,1} p_k^B\ H\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\det\rho_k^B}\right) \tag{37}$$

and can explicitly expressed as a function of θ and φ. Then, the minimization of $\widetilde{S}$ can performed over the polar and azimuthal angles. Nevertheless, there exists another elegant way to optimize the conditional entropy. It is based on the Koashi-Winter relation [26] (see also [19]) as we shall explain in what follows.

4.4 Minimization of conditional entropy

The Koashi-Winter relation establishes the connection between the classical correlation of a bipartite state ρ^{AB} and the entanglement of formation of its complement ρ^{BC}. This connection requires the purification of the state ρ^{AB}. In this respect, as ρ^{AB} is a two-qubit state of rank two, it decomposes as

$$\rho^{AB} = \lambda_1|\psi_1\rangle\langle\psi_1| + \lambda_2|\psi_2\rangle\langle\psi_2| \tag{38}$$

where the eigenvalues λ_1 and λ_2 are given by (30) and the eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ are

$$|\psi_1\rangle = \frac{1}{\sqrt{a_\alpha^2 a_{\alpha t}^2 + b_\alpha^2 b_{\alpha t}^2}}(a_\alpha a_{\alpha t}|u_\alpha, u_{\alpha t}\rangle + b_\alpha b_{\alpha t}|v_\alpha, v_{\alpha t}\rangle)$$
$$|\psi_2\rangle = \frac{1}{\sqrt{a_\alpha^2 b_{\alpha t}^2 + b_\alpha^2 a_{\alpha t}^2}}(a_\alpha b_{\alpha t}|u_\alpha, v_{\alpha t}\rangle + b_\alpha a_{\alpha t}|v_\alpha, v_{\alpha t}\rangle). \tag{39}$$

Attaching a qubit C to the bipartite system AB, we write the purification of ρ^{AB} as

$$|\psi\rangle = \sqrt{\lambda_1}|\psi_1\rangle \otimes |u_\alpha\rangle + \sqrt{\lambda_2}|\psi_2\rangle \otimes |v_\alpha\rangle \tag{40}$$

such that the whole system ABC is described by the pure density state $\rho^{ABC} = |\psi\rangle\langle\psi|$ from which one has the bipartite densities $\rho^{AB} = \mathrm{Tr}_C\rho^{ABC}$ and $\rho^{BC} = \mathrm{Tr}_A\rho^{ABC}$. Suppose now that a von Neumann measurement $\{M_0, M_1\}$ is performed on qubit A (here also we need positive operator valued measurement of rank one that is proportional the one dimensional projector). From the viewpoint of the whole system in the pure state $|\psi\rangle$, the measurement gives rise to an ensemble for ρ^{BC} that we denote by

$$\mathcal{E}^{BC} = \{p_k, |\phi_k^{BC}\rangle\}$$

where

$$p_k = \langle\psi|M_k \otimes \mathbb{I} \otimes \mathbb{I}|\psi\rangle \qquad |\phi_k^{BC}\rangle\langle\phi_k^{BC}| = \frac{1}{p_k}\mathrm{Tr}_A\Big[(M_k \otimes \mathbb{I} \otimes \mathbb{I})|\psi\rangle\langle\psi|\Big].$$

On the other hand, from the viewpoint of the state ρ^{AB}, the von Neuman measurement on A gives rise to the ensemble for ρ^B defined previously as $\mathcal{E}^B = \{p_k^B, \rho_k^B\}$. It is simple to check that the ensemble $\mathcal{E}^B$ can be induced from $\mathcal{E}^{BC}$ by tracing out the qubit C, namely

$$\rho_k^B = \mathrm{Tr}_C\Big[|\phi_k^{BC}\rangle\langle\phi_k^{BC}|\Big].$$

We denote by $E(|\phi_k^{BC}\rangle)$ the measure of entanglement for pure states. It is given by the von Neumann entropy of the reduced subsystem $\rho_k^B = \mathrm{Tr}_C(|\phi_k^{BC}\rangle\langle\phi_k^{BC}|)$

$$E(|\phi_k^{BC}\rangle) = S(\rho_k^B).$$

It follows that the average of entanglement of formation over the ensemble $\mathcal{E}^{BC}$

$$\overline{E}^{BC} = \sum_{k=0,1} p_k E(|\phi_k^{BC}\rangle)$$

coincides with the conditional entropy (5).At this level, it is important to notice that Koachi and Winter [26] have pointed out that the minimum value of $\overline{E}^{BC}$ is exactly the entanglement of formation of ρ^{BC}. Consequently,

$$\widetilde{S}_{\min} = E(\rho^{BC}). \tag{41}$$

This relation simplifies drastically the minimization process of the conditional entropy. Therefore, employing the prescription presented in [55] to get the concurrence of the density ρ^{BC}, one obtains

$$\widetilde{S}_{\min} = E(\rho^{BC}) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(\rho^{BC})|^2}\right) \tag{42}$$

with

$$C(\rho^{BC}) = \frac{\sqrt{p^2(1-p^{2r^2})(1-p^{2t^2})}}{(1+p^2)}.$$

It is remarkable that the minimal value of the conditional entropy given by (42) is reached for $\theta = \pi/2$ and $\varphi = 0$. Indeed, using the explicit form of the conditional entropy (37) in term of the polar and azimuthal angles θ and ϕ, one can verify that

$$\widetilde{S}_{\min} = \widetilde{S}(\theta = \frac{\pi}{2}, \phi = 0).$$

It is important to stress that the special values $\theta = 0$ and $\phi = \pi/2$ coincide with ones obtained in [18] (see also [56]) for optimal measurements to access quantum discord of two-qubit states. According to the equation (4), the classical correlation is

$$C(\rho^{AB}) = H\left(\frac{1}{2} + \frac{1}{2}\frac{p^{t^2} + p^{r^2+1}}{1+p^2}\right) - H\left(\frac{1}{2} + \frac{1}{2}\frac{\sqrt{1+p^2+p^{2r^2+2}+p^{2t^2+2}}}{1+p^2}\right) \tag{43}$$

and using the definition (6), the explicit expression of quantum discord reads

$$D(\rho^{AB}) = H\left(\frac{1}{2} + \frac{p}{1+p^2}\right) + H\left(\frac{1}{2} + \frac{1}{2}\frac{\sqrt{1+p^2+p^{2r^2+2}+p^{2t^2+2}}}{1+p^2}\right) - H\left(\frac{1}{2} + \frac{1}{2}\frac{p^{r^2} + p^{t^2+1}}{1+p^2}\right). \tag{44}$$

Note that for $r = 0$, the density state ρ^{AB} (23) reduces to the pure density Bell cat-states (22) and the the quantum discord (44) gives

$$D(|B_{\alpha,\alpha}\rangle\langle B_{\alpha,\alpha}|) = H\left(\frac{1}{2} + \frac{p}{1+p^2}\right) \tag{45}$$

which is, as expected, exactly the entanglement of formation $E(|B_{\alpha,\alpha}\rangle\langle B_{\alpha,\alpha}|)$. Indeed, it is simple to check that the concurrence of the pure Bell cat-state $|B_{\alpha,\alpha}\rangle$ is

$$C(|B_{\alpha,\alpha}\rangle) = \frac{1-p^2}{1+p^2} \tag{46}$$

which coincides for $r = 0$ with the concurrence of the state (23) given by

$$C(\rho^{AB}) = \frac{p^{r^2}\sqrt{(1-p^2)(1-p^{2t^2})}}{1+p^2}. \tag{47}$$

where we used the definition of the concurrence for mixed states introduced in [55]. It is also simply verified from (44) that for $r = 1$, the quantum discord vanishes and the concurrence (47) is also zero.

The behavior of quantum discord (44) versus the overlapping p and the reflection coefficient r is plotted in the figure 1. As seen from this figure, for p fixed the quantum discord decreases as we increase the reflection parameter r of the beam splitter and the maximum is obtained in the limiting case $r = 0$. This indicates that the noisy channel, inducing the decoherence effects, renders the system less correlated and subsequently the quantum discord decreases. In other hand, for a fixed value of r, the quantum discord starts increasing, reaches a maximal value and decreases after. To see this behavior, we plot in the figure 2, the quantum discord versus the overlapping for Bell cat-states for different values of the reflection coefficient r. In particular, under the action of a 50:50 beam splitter, the maximal value of quantum discord is reached for $p \simeq 0.4$. Clearly, for $r = 0$, the maximum of the quantum discord or entanglement of formation is attained for $p = 1$. From the figure 2, it is easily seen that the value of the overlap p, for which the quantum discord D is maximal, increases for increasing values of r.

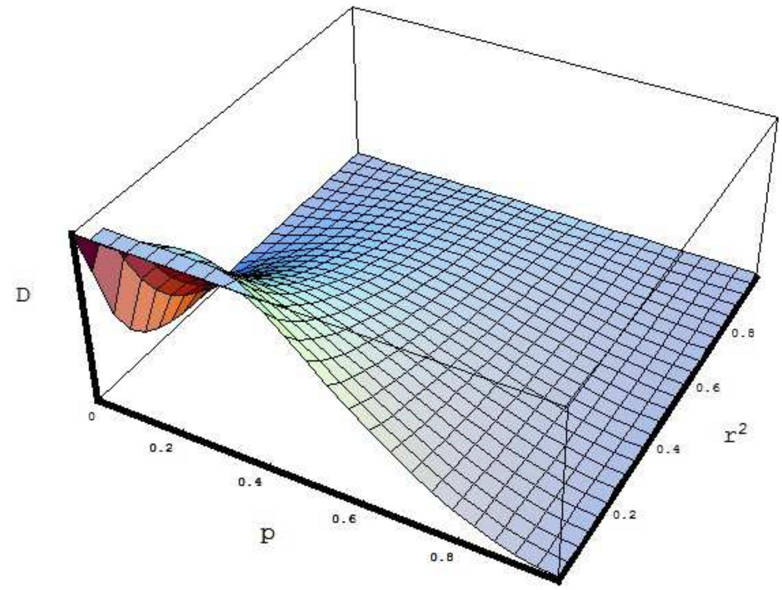

FIG. 1: The pairwise quantum discord D versus the overlapping p of Glauber states and the reflection parameter r^2.

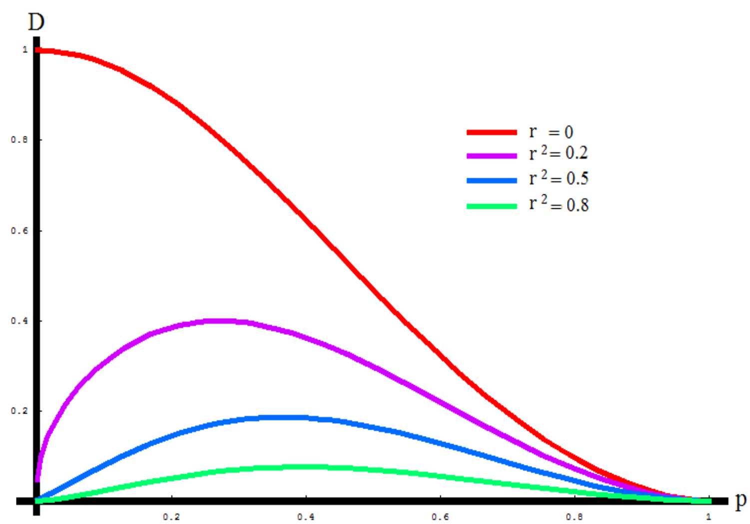

FIG. 2: The pairwise quantum discord D versus the overlapping p for different values of r

5 Evolution of quantum correlations under dephasing channel

In this section, we investigate the dynamics of bipartite quantum correlations (entanglement and quantum discord) of damped Bell cat-states, given by the density ρ^{AB} (26), under dephasing channel. To describe conveniently the effect of this channel, we use the the Kraus operator approach (see for instance [5]). In this approach, the time evolution of the bipartite density $\rho^{AB} \equiv \rho^{AB}(0)$ (26) can be written compactly as

$$\rho^{AB}(t) = \sum_{\mu,\nu} E_{\mu,\nu}(t)\ \rho^{AB}(0)\ E^{\dagger}_{\mu,\nu}(t)$$

where the so-called Kraus operators

$$E_{\mu,\nu}(t) = E_\mu(t) \otimes E_\nu(t) \qquad \sum_{\mu,\nu} E^\dagger_{\mu,\nu} E_{\mu,\nu} = \mathbb{I}.$$

The operators E_μ describe the one-qubit quantum channel effects. For a dephasing channel, the non-zero Kraus operators are given by

$$E_0 = \mathrm{diag}(1, \sqrt{1-\gamma}) \qquad E_1 = \mathrm{diag}(1, \sqrt{\gamma})$$

with $\gamma = 1 - e^{-\Gamma t}$ and Γ denoting the decay rate. This gives

$$\rho^{AB}(t) = \frac{2}{N_\alpha} \begin{pmatrix} (1+c)a_\alpha^2 a_{\alpha t}^2 & 0 & 0 & e^{-\Gamma t}(1+c)a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} \\ 0 & (1-c)a_\alpha^2 b_{\alpha t}^2 & e^{-\Gamma t}(1-c)a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} & 0 \\ 0 & e^{-\Gamma t}(1-c)a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} & (1-c)b_\alpha^2 a_{\alpha t}^2 & 0 \\ e^{-\Gamma t}(1+c)a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} & 0 & 0 & (1+c)b_\alpha^2 b_{\alpha t}^2 \end{pmatrix}. \quad (4$$

The entanglement in this state is measured by the concurrence

$$C = 2\ \max\{0, \Lambda_1(t), \Lambda_2(t)\}$$

where

$$\Lambda_1(t) = \frac{2}{N_\alpha} a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} \Big[(1-\gamma)(1+c) - (1-c)\Big] \qquad \Lambda_2(t) = \frac{2}{N_\alpha} a_\alpha a_{\alpha t} b_\alpha b_{\alpha t} \Big[(1-\gamma)(1-c) - (1+c)\Big].$$

Since $\Lambda_2(t)$ is non positive, the concurrence is

$$C(t) = \frac{1}{2} \frac{\sqrt{(1-p^2)(1-p^{2t^2})}}{1+p^2} \Big[e^{-\Gamma t}(1+p^{r^2}) - (1-p^{r^2})\Big] \tag{49}$$

for

$$t < t_0 = \frac{1}{\Gamma}[\ \ln(1+p^{r^2}) - \ln(1-p^{r^2})].$$

In this case, the system is entangled. However, for $t \geq t_0$, the concurrence is zero and the entanglement disappears., i.e. the system is separable. This shows clearly that under dephasing channel, the entanglement suddenly vanishes. Note that the bipartite system under consideration is entangled in the absence of external noise. Indeed, for $t = 0$, the concurrence (49) reproduces (47) which is non zero except in the limiting case $p \longrightarrow 0$ ($|\alpha| \longrightarrow +\infty$) or $r = 1$. The phenomenon of total loss of entanglement, termed in the literature "entanglement sudden death" [57], was experimentally confirmed under some specific conditions [58]. As the concurrence vanishes after a finite time t_0, it is interesting to ask what happens to quantum discord. The explicit expression of the amount quantum discord in the state $\rho^{AB}(t)$ can be obtained following the method described in the previous section. But, here we need only to know if the state $\rho^{AB}(t)$ has vanishing quantum discord. As the density $\rho^{AB}(t)$ has also the form of the letter X, one can use the criteria, classifying the so-called X states with vanishing quantum discord, discussed in [59]. Therefore, according to this criteria, $\rho^{AB}(t)$ has zero quantum discord if and only if $p \longrightarrow 0$. The quantum discord in the state $\rho^{AB}(t)$ is in general nonzero even when entanglement suddenly disappears. This gives a special instance of separable quantum states for which the quantum discord is non zero. This agrees with the commonly accepted fact that the quantum discord is a kind of quantum correlations which goes beyond entanglement and almost all quantum states have non vanishing quantum correlations [60].

6 Concluding remarks

To close this paper, let us briefly summarize the main results. Our effort was devoted to investigate the evolution of quantum discord of Bell cat-states under amplitude damping channel. We have derived the analytical expressions for the classical correlation and quantum discord. We discussed the usefulness of the Koashi-Winter relation to determine a closed form of quantum discord present in the system. We also discussed the evolution of the transmitted Bell cat-states under a dephasing channel. This provides a special instance for separable mixed quantum states with non vanishing quantum discord. The method used in this paper, to derive quantum discord, constitutes an alternative way to compute analytically classical correlations and quantum discord. It can be applied easily in evaluating the quantum discord present in the other Bell cat-states:

$$|B^{-}_{\alpha,\alpha}\rangle \sim |\alpha,\alpha\rangle - |-\alpha,-\alpha\rangle$$

$$|B^{+}_{\alpha,-\alpha}\rangle \sim |\alpha,-\alpha\rangle + |-\alpha,\alpha\rangle$$

$$|B^{-}_{\alpha,-\alpha}\rangle \sim |\alpha,-\alpha\rangle - |-\alpha,\alpha\rangle$$

after passing trough a amplitude damping channel. In addition, the analysis presented here for the evaluation of quantum discord is readily extended to more general systems, including squeezed states, $SU(2)$ and $SU(1,1)$ coherent states and so on. Finally, it will be interesting to compare the quantum discord in bipartite coherent states with its geometrized version usually called in the literature geometric quantum discord [61]. Further thought in this direction might be worthwhile.

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